Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 2, 1996

NONSTATIONARY NONISENTROPIC SPATIAL DOUBLE WAVE FLOWS

S. V. Meleshko

UDC 517.944 + 519.46

Nonisentropic nonstationary spatial double waves have been studied in [1–5], where some partial solutions of double wave equations were derived.

This paper classifies nonisentropic spatial double waves with an arbitrary equation of state $\tau = \tau(p, S)$ in the presence of functional arbitrariness in the general solution of Cauchy's problem, which cannot be reduced to invariant solutions.

Consideration is given to nonstationary, nonisobaric, nonisentropic double waves which are irreducible to invariant solutions for the equations of motion of an ideal gas in the spatial case:

$$\frac{d\mathbf{V}}{dt} + \tau\nabla p = 0, \quad \frac{d\tau}{dt} - \tau \operatorname{div} \mathbf{V} = 0, \quad \frac{dS}{dt} = 0. \tag{1}$$

The equation of state $\tau = \tau(p, S)$ is assumed to display the characteristics $\tau_p \neq 0$ and $\tau_S \neq 0$. In this case $\mathbf{V} = (u_1, u_2, u_3)$ is the velocity; p is the pressure; S is the entropy; τ is the specific volume; $d/dt = \partial/\partial t + u_\alpha \partial/\partial x_\alpha$ (summation over the recurring Greek index is performed from 1 to 3, unless otherwise stipulated).

If p and S are functionally dependent on the solution of system (1), then $p = p(S)$, $\tau = \tau(S)$, and system (1) can be written in the form

$$\frac{d\mathbf{V}}{dt} + \nabla\varphi = 0, \quad \frac{d\varphi}{dt} = 0, \quad \operatorname{div} \mathbf{V} = 0, \tag{2}$$

where $\varphi = \varphi(S)$ are found from $\varphi'(S) = \tau(S)\, p'(S) \neq 0$.

As double wave parameters we choose φ and some function λ which is functionally independent of the wave. From the first two equations of system (2) we obtain

$$\varphi_{x_i} \partial u_j/\partial\lambda - \varphi_{x_j} \partial u_i/\partial\lambda = 0 \quad (i, j = 1, 2, 3, \quad i \neq j). \tag{3}$$

Continuing (3) D/Dt ($D/Dt = D_t + u_\alpha D_\alpha$) and substituting the derivatives $D\varphi_{x_i}/Dt = -\varphi_{x_\alpha}((\partial u_\alpha/\partial\lambda)(\partial\lambda/\partial x_i) + (\partial u_\alpha/\partial\varphi)(\partial\varphi/\partial x_i))$, $\varphi_{x_i} = -(\partial u_i/\partial\lambda)(d\lambda/dt)$ we have

$$\frac{\partial u_i}{\partial\lambda}\frac{\partial\lambda}{\partial x_j} - \frac{\partial u_j}{\partial\lambda}\frac{\partial\lambda}{\partial x_i} + \frac{d\lambda}{dt}\left(\frac{\partial^2 u_j}{\partial\lambda^2}\frac{\partial u_i}{\partial\lambda} - \frac{\partial^2 u_i}{\partial\lambda^2}\frac{\partial u_j}{\partial\lambda}\right) = 0 \quad (i, j = 1, 2, 3, \quad i \neq j). \tag{4}$$

As follows from the prohibition of reduction to invariant solutions [6], the rank of the matrix composed of the coefficients for the derivatives $d\lambda/dt$, $\partial\lambda/\partial x_i$ $(i, j = 1, 2, 3, i \neq j)$ in Eqs. (4) and the fifth equation of system (2) must be less than or equal to 2. Hence we arrive at the equation $\partial u_i/\partial\lambda = 0$ $(i = 1, 2, 3)$ which contradicts the nonisentropicity of the flow. Thus, flows should be considered for which p and S are functionally independent.

We choose pressure and entropy as double wave parameters, i.e., assume that $u_i = u_i(p, S)$ $(i = 1, 2, 3)$. Introducing the new dependent variable $\varphi = (\operatorname{div}\mathbf{V})/\tau_p$, we reduce system (1) to the form ($H = \tau_p + u_{\alpha p}u_{\alpha p}$):

$$\frac{dp}{dt} - \tau\varphi = 0, \quad S_0 = \frac{dS}{dt} = 0, \quad R_1 = u_{\alpha S}S_{x_\alpha} - H\varphi = 0, \quad \Phi_i = p_{x_i} + u_{ip}\varphi = 0 \quad (i = 1, 2, 3). \tag{5}$$

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 2, pp. 50–61, March–April, 1996. Original article submitted January 1, 1995.

Differentiating D_i completely with respect to the spatial variable x_i and deriving the combinations below, from Eqs. (5), we obtain

$$D_i\Phi_j - D_j\Phi_i = u_{jp}\varphi_{x_i} - u_{ip}\varphi_{x_j} + (u_{jpS}S_{x_i} - u_{ipS}S_{x_j}) - \varphi^2 (u_{jpp}u_{ip} - u_{ipp}u_{jp}) = 0; \tag{6}$$

$$R_2 = \frac{D(H\varphi - u_{\alpha S}S_{x_\alpha})}{Dt} = H\frac{d\varphi}{dt} - \varphi(\tau u_{\alpha pS} + u_{\alpha p}u_{\beta p}u_{\beta S})S_{x_\alpha} + \varphi^2(H^2 + \tau H_p) = 0; \tag{7}$$

$$D(\Phi_i)/Dt = u_{ip}d\varphi/dt + \tau\varphi_{x_i} + \varphi\zeta S_{x_i} - \varphi^2(u_{ip}H - \tau u_{ipp}) = 0 \quad (i, j = 1, 2, 3, \ i \neq j), \tag{8}$$

where $\zeta = \tau_S + u_{\alpha p}u_{\alpha S}$; $\xi = \tau_S + 2u_{\alpha p}u_{\alpha S}$; $D/Dt = D_t + u_\alpha D_\alpha$.

Eliminating the derivatives φ_{x_i} and φ_{x_j} from (6) and using Eqs. (8), we get

$$(\zeta u_{ip} - \tau u_{ipS})S_{x_j} - (\zeta u_{jp} - \tau u_{jpS})S_{x_i} = 0 \quad (i, j = 1, 2, 3, \ i \neq j). \tag{9}$$

Further, as in the case of stationary and plane flows, it is necessary to distinguish between two cases: $H \neq 0$ and $H = 0$.

(1) Let $H \neq 0$. Expressing φ in terms of the third equation of system (5) and substituting it into the other equations of the system, together with (9) we derive a homogeneous system of seven quasilinear differential equations with respect to p and S. From the prohibition of double wave reduction to invariant solutions [6] it follows that

$$\tau u_{ipS} - \zeta u_{ip} = 0 \quad (i = 1, 2, 3). \tag{10}$$

If $u_{ip} = 0 \quad (i = 1, 2, 3)$, then

$$p = h(t), \quad \varphi = h'/\tau, \quad S_t + u_\alpha(S)S_{x_\alpha} = 0, \tag{11}$$

where $u_i(S)$ are arbitrary functions; $h(t)$ is a function satisfying the equation

$$h'' = -(h')^2 \partial \ln(|\tau_p|)/\partial p. \tag{12}$$

Since p and S are functionally independent, the last equality gives $\partial^2 \ln(|\tau_p|)/\partial p \partial S = 0$ and hence,

$$\tau = A_1(S)g(p) + A_2(S) \tag{13}$$

[$A_1(S)$, $A_2(S)$, and $g(p)$ are arbitrary functions]. Substituting the latter expression for τ into (12) and integrating it with respect to the variable t, we obtain $g(h(t)) = c_1 t + c_2$ (c_1 and c_2 are arbitrary constants, $c_1 \neq 0$). Without loss of generality it is assumed that $c_2 = 0$.

Thus, for an equation of state of type (13), double waves exist in which $u_i(S)$ are arbitrary functions of entropy; the pressure p is determined from the equation $g(p) = c_1 t$, and the entropy S satisfies the following system of two differential equations in partial derivatives:

$$dS/dt = 0, \quad u'_\alpha S_{x_\alpha} = c_1 A_1/(c_1 t A_1 + A_2). \tag{14}$$

System (14) is in involution and contains one arbitrary function of two arguments. For instance, for $A_2 = 0$ its solution is

$$tu_1(S) - x_1 + \psi(tu_2(S) - x_2, \, tu_3(S) - x_3) = 0$$

[$\psi(\xi, \zeta)$ is an arbitrary function].

We now consider the case where $u_{\alpha p}u_{\alpha p} \neq 0$ (for definiteness it is assumed that $u_{1p} \neq 0$). From (10), the existence of the functions $F_i = F_i(p)$ $(i = 0, 2, 3)$ follows. Hence,

$$\tau = F_0 u_{1p}^2 - u_\alpha u_{\alpha p}, \quad u_{ip} = F_i u_{1p} \quad (i = 2, 3). \tag{15}$$

Excluding the derivatives $d\varphi/dt$ from Eqs. (8) and using Eq. (7) we get

$$\Psi_i = \tau H\varphi_{x_i} + \varphi(H\zeta S_{x_i} + u_{ip}\xi u_{\alpha p}S_{x_\alpha}) - \varphi^2 c_i = 0, \quad c_i = 2u_{ip}H^2 + \tau H_p u_{ip} - \tau H u_{ipp} \quad (i = 1, 2, 3). \tag{16}$$

From the combination

$$\tau\varphi\xi u_{\alpha p} D_\alpha R_1 - u_{\alpha S}(D\Psi_\alpha/Dt - \tau D_\alpha R_2) + \varphi(\xi u_{\alpha p}u_{\alpha S} + \zeta H)u_{\beta p}D_\beta S_0 = 0$$

we find the first-order equation for the dependent variables:

$$R_3 = H(u_{\alpha p}S_{x_\alpha})[\tau\xi(u_{\beta SS}S_{x_\beta}) + \varphi(2H\xi u_{\alpha p}u_{\alpha S} - 2\xi\tau u_{\alpha pp}u_{\alpha S} - H\tau\xi_S)] + \varphi^2 b = 0. \tag{17}$$

The form of the function $b = b(p, S)$ is rather tedious and will not be used in further discussions.

With $u_{\alpha p}S_{x_\alpha} = 0$ a double wave exists only for $u_i = u_i(S)$ $(i = 2, 3)$. Indeed, differentiating the last equation along the particle trajectory and allowing for conservation of entropy in the particle, we obtain $D(u_{\alpha p}S_{x_\alpha})/Dt = \tau\varphi u_{\alpha pp}S_{x_\alpha} = 0$. Thus, the prohibition of reduction to the invariant solution makes it necessary to fulfill the following:

$$\operatorname{rang}\begin{Vmatrix} u_{1p} & u_{2p} & u_{3p} \\ u_{1pp} & u_{2pp} & u_{3pp} \end{Vmatrix} \leqslant 1.$$

In this case, $F_2' = F_3' = 0$ and, rotating the axis of coordinates, one get $u_i = u_i(S)$ $(i = 2, 3)$. In this case, the equation $u_{\alpha p}S_{x_\alpha} = 0$ gives $S_{x_1} = 0$ and from the relation $D_1R_1 = 0$ we obtain $\varphi_{x_1} = H_p u_{1p}\varphi^2/H$. Substituting the latter into (16) $(i = 1)$, we get $2u_{1p}H - \tau u_{1pp} = 0$. Hence, from $\tau = u_{1p}(F_0 - u_1)$ and (15), it follows that

$$2u_{1p}F_0' + u_{1pp}(F_0 - u_1) = 0.$$

In this case the system of equations (5), (7), and (16) along with $u_{\alpha p}S_{x_\alpha} = 0$ are in involution and contain one arbitrary function of one argument.

We consider the case where $u_{\alpha p}S_{x_\alpha} \neq 0$. Assume that $\xi \neq 0$. Let us demonstrate that in this case the system of equations (5) has no solutions with functional arbitrariness in the general solution of Cauchy's problem.

If

$$r = \operatorname{rang}\begin{Vmatrix} u_{1p} & u_{2p} & u_{3p} \\ u_{1S} & u_{2S} & u_{3S} \end{Vmatrix} = 1,$$

either reduction to the invariant solution or contradiction with the condition $H \neq 0$ occurs. Thus, it is assumed that $r = 2$.

We introduce now the new dependent variable $\lambda = u_{\alpha p}S_{x_\alpha}$ [in terms of compatibility theory, the space of dependent variables is partially extended, i.e., we consider the extended space not in its entirety but only its subspace].

Since $\xi \neq 0$, it follows from the equations

$$R_{3+i} = -\tau D_i R_2 + D\Psi_i/Dt - \varphi\xi u_{ip}u_{\alpha p}D_\alpha S_0 - \varphi\zeta H D_i S_0 = 0$$

that

$$L_i = \lambda_{x_i} - a_{i\alpha}(p, S)S_{x_\alpha} - c(p, S, \lambda, \varphi) = 0 \quad (i = 1, 2, 3)$$

with the functions $a_{ij} = a_{ij}(p, S)$, $c = c(p, S, \lambda, \varphi)$ $(i, j = 1, 2, 3)$. In addition,

$$L_0 = \frac{D}{Dt}(\lambda - u_{\alpha p}S_{x_\alpha}) = \frac{d\lambda}{dt} - \tau\varphi u_{\alpha pp}S_{x_\alpha} + \tau_p\varphi\lambda = 0.$$

For the dependent variables p, S, φ, and λ the overdetermined system of differential equations is deduced in which the parametric derivative is either S_{x_2} or S_{x_3}.

We determine the coefficients of the second derivatives in the following extensions of the equations $(i = 1, 2, 3)$:

$$D_iR_1 = u_{\alpha S}S_{x_\alpha x_i} + \ldots = 0, \quad D_iR_3 = \tau\lambda H\xi u_{\alpha SS}S_{x_\alpha x_i} + \ldots = 0,$$

$$\frac{DL_i}{Dt} - D_iL_0 + a_{i\alpha}D_\alpha S_0 = \tau\varphi u_{\alpha pp}S_{x_\alpha x_i} + \ldots = 0, \quad D_i(\lambda - u_{\alpha p}S_{x_\alpha}) = u_{\alpha p}S_{x_\alpha x_i} + \ldots = 0.$$

In this case only terms containing the second derivatives are written.

Since $H\,\lambda\,\varphi\,\xi \neq 0$, and the parameteric derivatives can be represented as $S_{x_i x_j}$ $(i,\, j = 1,\, 2,\, 3)$ only, the solution having functional arbitrariness requires that in this system the matrix rank be notequal to the number of these derivatives. Hence,

$$r_0 = \operatorname{rang}\begin{Vmatrix} u_{1p} & u_{2p} & u_{3p} \\ u_{1S} & u_{2S} & u_{3S} \\ u_{1SS} & u_{2SS} & u_{3SS} \\ u_{1pp} & u_{2pp} & u_{3pp} \end{Vmatrix} \leqslant 2.$$

If

$$\begin{Vmatrix} u_{1p} & u_{2p} & u_{3p} \\ u_{1S} & u_{2S} & u_{3S} \\ u_{1pp} & u_{2pp} & u_{3pp} \end{Vmatrix} = 0,$$

after integration over S, it acquires the form

$$(F_2'F_3 - F_3'F_2)\, u_1 + F_3'u_2 - F_2'u_3 = g\,(p) \tag{18}$$

with the arbitrary function $g\,(p)$. Differentiating the last equation with respect to p we get

$$(F_2''F_3 - F_3''F_2)\, u_1 + F_3''u_2 - F_2''u_3 = g'\,(p). \tag{19}$$

From (18) and (19), we obtain the relations

$$(F_2'F_3'' - F_3'F_2'')(u_3 - F_3u_1) = F_3'g'(p) - F_3''g(p), \quad (F_2'F_3'' - F_3'F_2'')(u_2 - F_2u_1) = F_2'g'(p) - F_2''g(p). \tag{20}$$

When $F_2'F_3'' - F_3'F_2'' \neq 0$, from Eq. (20) we find $u_i = u_i\,(p)$ $(i = 1,\, 2,\, 3)$, which in view of the equation $R_1 = 0$ contradicts the condition $H\varphi \neq 0$. Hence, $F_2'F_3'' - F_3'F_2'' = 0$. Without loss of generality, it is assumed that $F_3 = 0$. In this case either $F_2' = 0$ or $u_{3S} = 0$. In both cases, considering the relation $r_0 = 0$ leads to reduction (plane flows).

Thus, we further study the case with $\xi = 0$. From (10) and $\xi = 0$, it follows that $\tau = f_0 u_{1p}^2$ with $f_0 = f_0\,(p)$. From the combinations

$$D_{x_i}\Psi_j - D_{x_j}\Psi_i = 0 \quad (i = 1,\, 2,\, 3,\ i \neq j),$$

we deduce the first-order equations

$$\begin{gathered} S_{x_i}a_j - S_{x_j}a_i = -\varphi\,(u_{ip}d_j - u_{jp}d_i) \quad (i,\, j = 1,\, 2,\, 3,\ i \neq j), \\ a_1 = \tau H u_{1p}\,[(H_p/H)_S - 2\,(u_{1pp}/u_{1p})_S + 2\,(H/\tau)_S], \\ a_i = F_i a_1, \quad d_i = F_i d_1 + F_i'\,(u_{1p}\,(2H^2 + \tau H_p) - 2\tau H u_{1pp}) - \tau H u_{1p} F_i'' \quad (i = 2,\, 3). \end{gathered} \tag{21}$$

As follows from (21) and $R_1 = 0$, the relations $a_i\,(u_{\alpha S}a_\alpha) = 0$ $(i = 1,\, 2,\, 3)$ must be satisfied for the solution to have functional arbitrariness. Since $\tau_S \neq 0$, it follows that $a_1 = 0$ and from (21) it follows that $d_i = F_i d_1$ $(i = 2,\, 3)$. Hence,

$$(H_p/H)_S - 2\,(u_{1pp}/u_{1p})_S + 2\,(H/\tau)_S = 0; \tag{22}$$

$$F_i'\,(u_{1p}\,(2H^2 + \tau H_p) - 2\tau H u_{1pp}) - \tau H u_{1p} F_i'' = 0 \quad (i = 2,\, 3). \tag{23}$$

From the equations

$$D\Psi_i/Dt - \tau D_i R_2 + \varphi H\, u_{\alpha p}\, u_{\alpha S} D_i\, S_0 = 0 \quad (i = 1,\, 2,\, 3)$$

on the strength of (22), the following relation follows:

$$2\tau^2 H u_{1ppp} - 2\tau u_{1pp}\,(2H^2 + \tau H_p) + u_{1p} H\,(4H^2 - \tau H_p + \tau\tau_{pp}) = 0. \tag{24}$$

Equations (22)–(24) and

$$\xi = \tau_S + 2\, u_{\alpha p}\, u_{\alpha S} = 0 \tag{25}$$

are sufficient for the involutivity of the overdetermined system of equations (5), (7), and (16) with two arbitrary functions of one argument. The study of the compatibility of the system of equations (22)–(25) involves two stages: $F_2' = F_3' = 0$ and $(F_2')^2 + (F_3')^2 \neq 0$.

(A) Let $(F_2')^2 + (F_3')^2 \neq 0$. Since $\tau H u_{1p} \neq 0$, from (23) it follows that $(F_2' F_3'' - F_3' F_2'') = 0$. Without loss of generality, $F_2' \neq 0$, $F_3 = 0$, and $u_3 = u_3(S)$ are assumed to be arbitrary functions. Hence, Eq. (22) is identically fulfilled in view of (23) ($i = 2$).

From Eqs. (15) and (25), and $\tau = f_0 u_{1p}^2$ we obtain the first derivatives

$$u_{2S} = -(f_0\, u_{1pS} + u_{1S})/F_2, \quad u_{2p} = F_2\, u_{1p}. \tag{26}$$

Equating the mixed derivatives and integrating over S we determine

$$u_{1pp} = -(u_{1p}\,(F_2 f_0' - f_0 F_2' + F_2\,(1 + F_2^2)) - u_1 F_2' + F_4)/(f_0 F_2),$$

where $F_4 = F_4(p)$ is an arbitrary function. Since the calculations are too cumbersome, we just indicate the way to further study this problem. Differentiating u_{1pp} with respect to p, we find u_{1ppp}. Substituting the derivatives u_{1pp} and u_{1ppp} into (23) and (24), we get $F_4 = cF_2'$ ($c = \text{const}$), and for $u_1 = u_1(p, S)$, we derive the equation

$$a u_{1p} + (u_1 - c) = 0$$

with the function $a = a(p)$. Without loss of generality, it is assumed that $c = 0$. From the last equation we find $u_1 = h_1(p)\, A(S)$. Then, from (26) we obtain $u_2 = h_2(p)\, A(S)$ and from the condition $\tau = f_0 u_{1p}^2$ it follows that the equation of state can be represented in the form $\tau = g(p)\, A^2(S)$. In this case, $A = A(S)$ is an arbitrary function and for the function $g = g(p)$ and $h_i = h_i(p)$ ($i = 1, 2$) two ordinary differential equations hold:

$$F_2 F_2''\,((h_1')^2\,(1 + F_2^2) + g') = 2\,(F_2')^2 h_1'\,(g' h_1 - h_1' g)$$
$$+ F_2 F_2'\, g\, g'' + 2\,((F_2')^2 g + (h_1')^2 F_2 F_2'\,(1 + F_2^2))\,((h_1')^2\,(1 + F_2^2) + g'); \tag{27}$$

$$F_2\, g\, h_1'' = F_2\,(1 + F_2^2)\,(h_1')^3 - F_2' h_1\,(h_1')^2 + h_1'\,(F_2\, g' - g\, F_2') \tag{28}$$

($F_2 = h_2'/h_1'$). Here, as $\tau + u_\alpha u_{\alpha p} = 0$, the pressure is also independent of time t. The function $u_3 = u_3(S)$ remains arbitrary.

Remark 1. In [1–3], for a polytropic gas with the function $u_3 = c\,A(S)$ ($c = \text{const}$), another representation of the solution deduced here is given. Thus, if the solution is sought for in the form $u_1^2 = \psi(p)\, A^2(S)$, for the functions $F_2(p)$ and $\psi(p)$ there is a system of two ordinary differential equations

$$g\,(2\psi\psi'' - (\psi')^2)/(2\psi\psi') = g' + (1 + F_2^2)\,(\psi')^2/(4\psi) - (g + \psi'/2)\, F_2'/F_2; \tag{29}$$

$$g\,(g' + (1 + F_2^2)\,(\psi')^2/(4\psi)) = (1 + F_2^2)^2 (\psi')^2/(8\psi^2)$$
$$+ (g'\,(1 + F_2^2) + g F_2 F_2')\,\psi'/(2\psi) + \psi'\, g'\, F_2'/F_2 + g\,(g'' + 2\,g'\, F_2'/F_2). \tag{30}$$

Remark 2. In stationary spatial double waves, the equation $\tau + u_\alpha u_{\alpha p} = 0$ provides identical fulfillment of Eq. (24) by virtue of (23) with $(F_2')^2 + (F_3')^2 \neq 0$.

(B) Let $F_2' = F_3' = 0$. Without loss of generality, it is assumed that $F_2 = F_3 = 0$, which corresponds to $u_2 = u_2(S)$, $u_3 = u_3(S)$. Then, from the condition $\xi = 0$, we obtain $u_1 = h_1(p)\, A(S) + \psi(p)$, and from $\tau = f_0 u_{1p}^2$ it follows that $f_0 = -h_1/h_1'$. Analyzing Eqs. (22) and (24) (without loss of generality) we arrive at

$$u_1 = h_1(p)\, A(S), \qquad \tau = g(p)\, A^2(S), \tag{31}$$

where $g = -h_1 h_1'$. In this case, the functions $u_2(S)$ and $u_3(S)$ remain arbitrary.

In [1–3] these functions are related to $A(S)$ due to the additional requirement upon the double wave shape.

(2) Let $H = 0$. Since p and S are functionally independent, from the prohibition of reduction to the

invariant solutions of systems (5), (7), and (9) it follows that

$$\operatorname{rang}\left\|\begin{matrix} u_{1S} & u_{2S} & u_{3S} \\ a_1 & a_2 & a_3 \\ b_2 & -b_1 & 0 \\ b_3 & 0 & -b_1 \\ b_3 & 0 & -b_1 \end{matrix}\right\| \leqslant 2,$$

where $b_i = \tau u_{ipS} - \zeta u_{ip}$; $a_i = b_i - \xi u_{ip}$ $(i = 1, 2, 3)$. Therefore, with $b_\alpha b_\alpha \neq 0$ the following relations must be satisfied:

$$b_\alpha a_\alpha = 0, \qquad u_{\alpha S} b_\alpha = 0. \tag{32}$$

We further pass to the new independent variables p, S, x_3, and t (without loss of generality the relationship $p_{x_1} S_{x_2} - p_{x_2} S_{x_1} \neq 0$ is assumed to be fulfilled), i.e., $x_1 = P(p, S, x_3, t)$, $x_2 = Q(p, S, x_3, t)$.

Thus, Eqs. (1) can be written as

$$\begin{gathered} BP_p - AQ_p = 0, \quad -u_{1p} BP_S + (\tau + u_{1p} A) Q_S = 0, \quad (\tau + u_{2p} B) P_S - u_{2p} AQ_S = 0, \\ (u_{3p} B - \tau Q_{x_3}) P_S - (u_{3p} A - \tau P_{x_3}) Q_S = 0, \\ (u_{2S} - u_{3S} Q_{x_3}) P_p - (u_{1S} - u_{3S} P_{x_3}) Q_p = 0, \\ A = u_1 - u_3 P_{x_3} - P_t, \quad B = u_2 - u_3 Q_{x_3} - Q_i. \end{gathered} \tag{33}$$

Hence,

$$P_p Q_S - P_S Q_p \neq 0. \tag{34}$$

The study of system (33) consists of two cases:

(a) there are values of i and j $(i \neq j)$ such that $u_{ipS} u_{jp} - u_{jpS} u_{ip} \neq 0$,

(b) $u_{ipS} u_{jp} - u_{jpS} u_{ip} = 0$ $(i, j = 1, 2, 3, \ i \neq j)$.

We now consider case (a). For definiteness it is assumed that $\Delta_3 = u_{2pS} u_{1p} - u_{1pS} u_{2p} \neq 0$. We introduce the notations

$$\Delta_1 = u_{3pS} u_{2p} - u_{2pS} u_{3p}, \quad \Delta_2 = u_{1pS} u_{3p} - u_{3pS} u_{1p}.$$

Then system (33) reduces to the system

$$P = \left(x_3 \Delta_1 + t \left\| \begin{matrix} \psi & u_{2p} \\ \psi_S & u_{2pS} \end{matrix} \right\| + \left\| \begin{matrix} \chi & u_{2p} \\ \chi_S & u_{2pS} \end{matrix} \right\| \right) / \Delta_3; \tag{35}$$

$$Q = \left(x_3 \Delta_2 - t \left\| \begin{matrix} \psi & u_{1p} \\ \psi_S & u_{1pS} \end{matrix} \right\| - \left\| \begin{matrix} \chi & u_{1p} \\ \chi_S & u_{1pS} \end{matrix} \right\| \right) / \Delta_3; \tag{36}$$

$$(u_2 - u_3 Q_{x_3} - Q_t) P_p - (u_1 - u_3 P_{x_3} - P_t) Q_p = 0 \tag{37}$$

$[\psi = \tau + u_\alpha u_{\alpha p}, \ \chi = \chi(p, S)]$.

Substituting (35) and (36) into (37) we get a relationship which is linear in x_3 and t. Splitting it with respect to x_3 and t, we obtain the following equations: for a free term, a linear second-order hyperbolic equation in function $\chi = \chi(p, S)$ of the type

$$\tau \chi_{pS} - \zeta \chi_p + q_1 \chi_S + q_2 \chi = 0 \tag{38}$$

where the functions $q_1 = q_1(p, S)$ and $q_2 = q_2(p, S)$ are expressed in terms of τ and $u_i(p, S)$ $(i = 1, 2, 3)$, and for x_3 and t,

$$\Delta_\alpha b_{\alpha p} = 0, \quad (u_{\alpha p} b_\alpha) \Delta_3 + b_1 b_{2p} - b_2 b_{1p} = 0. \tag{39}$$

Thus, at $H = 0$ the flows in the case (a) will have linear levels and the functions $\tau(p, S)$ and $u_i(p, S)$ $(i = 1, 2, 3)$ satisfy the overdetermined system consisting of five differential equations: (32), (39), and $H = 0$. It is rather difficult to analyze this system in the general case. However, assuming $\psi = 0$, the flows become stationary. For stationary gaseous flows with the equation of state of the form $\tau = g(p) A^2(S)$, this system is consistent only in a partricular case of the equation of state for a polytropic gas with polytropic index $\gamma = 2$ and contains one arbitrary function of one argument.

For case (b) without loss of generality it is assumed that $u_{1p} \neq 0$. Hence,

$$u_{ip} = F_i u_{1p} \quad (i = 2, 3), \tag{40}$$

where $F_i = F_i(p)$ $(i = 2, 3)$. In this case, the solution of system (33) by virtue of (34) reduces to the integration of a system of two linear equations with respect to one unknown function $Q(p, S, x_3, t)$:

$$\omega Q_p + \beta (u_3 Q_{x_3} + Q_t - u_2) = 0; \quad \omega_S Q_p + \beta (u_{3S} Q_{x_3} - u_{2S}) = 0. \tag{41}$$

In this case,

$$\begin{gathered} P = -F_2 Q - x_3 F_3 + t F_4 + F_0, \\ \omega = u_1 + u_2 F_2 + u_3 F_3 - F_4, \quad \beta = F_0' - F_2' Q - x_3 F_3' + t F_4', \quad \tau = -\omega u_{1p}; \end{gathered} \tag{42}$$

$F_0 = F_0(p)$, $F_4 = F_4(p)$ are arbitrary functions. Note that Eq. (42) gives

$$x_1 + x_2 F_2 + x_3 F_3 + t F_4 = F_0,$$

and substituting (42) into the reversibility condition (34) we obtain $\beta Q_S \neq 0$. In addition, since $\tau = -\omega u_{1p}$, we have $\omega \neq 0$ and $\xi = -\omega^2 (u_{1p}/\omega)_S$. Excluding Q_p from (41) by virtue of $\beta \neq 0$, we arrive at

$$(u_3/\omega)_S Q_{x_3} + (1/\omega)_S Q_t = (u_2/\omega)_S. \tag{43}$$

We demonstrate now that $\omega_S \neq 0$. Indeed, let $\omega_S = 0$; then $u_{3S} \neq 0$, and with $u_{3S} = 0$ from (43) we find $u_{2S} = 0$, and hence $u_{1S} = 0$, which contradicts the condition $\tau_S \neq 0$. Since $u_{3S} \neq 0$, relationship (43) is integrated over x_3:

$$Q = x_3 (u_{2S}/u_{3S}) + G(t, p, S).$$

Substituting this relation into the first equation of (41) and splitting it with respect to x_3, we get

$$\omega (u_{2S}/u_{3S})_p - (F_2' (u_{2S}/u_{3S}) + F_3')(u_3 (u_{2S}/u_{3S}) + G_t - u_2) = 0; \tag{44}$$

$$\omega G_p + (F_0' - F_2' G + t F_4')(u_3 (u_{2S}/u_{3S}) + G_t - u_2) = 0. \tag{45}$$

If $F_2' u_{2S} + F_3' u_{3S} = 0$, from Eq. (44) we find that $(u_{2S}/u_{3S})_p = 0$, which also leads to the contradictory equality $\tau_S = 0$.

Differentiating the relation $\tau + \omega u_{1p} = 0$ with respect to p using the condition $H = 0$, we deduce

$$F_2' u_2 + F_3' u_3 - F_4' = -\omega (u_{1pp}/u_{1p}). \tag{46}$$

Since $\omega_S = 0$, we have,

$$F_2' u_{2S} + F_3' u_{3S} = -\omega (u_{1pp}/u_{1p})_S.$$

On the other hand, $\omega_S = u_{1S} + u_{2S} F_2 + u_{3S} F_3 = 0$. Differentiating the latter with respect to p and using (40), we have $F_2' u_{2S} + F_3' u_{3S} = -u_{1pS}(1 + F_2^2 + F_3^2)$. However, in this case $(\tau u_{1p}/u_{1pS})_p = 0$. Since $\tau = -\omega u_{1p}$, it follows that

$$u_{1p} = 1/(\omega g + \lambda), \tag{47}$$

where $g = g(S)$; $\lambda = \lambda(p)$. In this case, $\tau = -\omega/(\omega g + \lambda)$ and hence $g'\lambda \neq 0$, and if $g'\lambda = 0$, then $\tau_S \tau_p = 0$. Substituting τ and u_{1p} into $H = 0$ we obtain $1 + F_2^2 + F_3^2 = \lambda^2 (\omega/\lambda)_p$. Thus, $(\omega/\lambda)_p \neq 0$. As $(F_2')^2 + (F_3')^2 \neq 0$, let, e.g., $F_3' \neq 0$. From (46) it follows that

$$u_3 = (-F_2' u_2 + F_4' - \omega (u_{1pp}/u_{1p}))/F_3'. \tag{48}$$

Differentiating (48) with respect to p and using (40), we get

$$u_2 (F_2'/F_3')' = -u_{1p}(F_2 F_2' + F_3 F_3')/F_3' + [F_4' - \omega (u_{1pp}/u_{1p})/F_3']_p. \tag{49}$$

When $(F_2'/F_3')' = 0$, it is assumed that $F_2 = 0$. Substituting (47) into (48) we obtain a quadratic polynomial with respect to $g(S)$. As $g' \neq 0$, the coefficients for g must go to zero. Hence, $(\omega/\lambda)_p = 0$ and this contradicts the above assumption. Thus $(F_2'/F_3')' \neq 0$; then from (49) we determine u_2 and calculations similar to those for $(F_2'/F_3')' = 0$ lead to the same contradiction. Hence, $\omega_S \neq 0$.

Integrating (44) over (46) we have the relation

$$Q = t\,\frac{(u_2/\omega)_S}{(1/\omega)_S} + G(p, S, \lambda) \quad \left(\lambda = x_3 - t\,\frac{(u_2/\omega)_S}{(1/\omega)_S}\right),$$

Substituting it into (45) and decomposing it with respect to t, we get

$$k_2 - k_3 G_\lambda = 0; \tag{50}$$

$$G_p - G_\lambda (F_0' - F_2'G - F_3'\lambda)\, u_{3S}/\omega_S = (F_0' - F_2'G - F_3'\lambda)\, u_{2S}/\omega_S, \tag{51}$$

where $k_i = F_i\omega_S\,(u_{1p}^2/\tau)_S + u_{iS}\,(\ln(\tau/u_{1p}^2))_S \quad (i = 2, 3)$.

We assume that $k_3 = 0$, then $k_2 = 0$. Hence,

$$F_2\,\omega_S\,(u_{1p}^2/\tau)_S + u_{2S}\,(\ln(\tau/u_{1p}^2))_S = 0, \quad F_3\,\omega_S\,(u_{1p}^2/\tau)_S + u_{3S}\,(\ln(\tau/u_{1p}^2))_S = 0. \tag{52}$$

Further, it is necessary to consider two cases:

1. Let $F_2u_{3S} - F_3u_{2S} \neq 0$. Then from (52) we find that $(u_{1p}^2/\tau)_S = 0$, i.e., $\tau = \varphi(p)\,u_{1p}^2$ and from $H = 0$ it follows that

$$\tau = g(p)\,A^2(S), \quad u_i = A(S)\,(h_i(p) + g_i(S)) \quad (i = 1, 2, 3).$$

Further analysis leads (without loss of generality) to the functions $g(p)$ and $h_i(p)$ $(i = 1, 2, 3)$ satisfying the equations

$$g' + h_\alpha' h_\alpha' = 0, \quad g + h_\alpha h_\alpha' = 0, \tag{53}$$

and the velocity coordinates are represented either as

$$u_i = A(S)\,h_i(p) \quad (i = 1, 2, 3)$$

or as

$$u_i = A(S)\,h_i(p) \quad (i = 1, 2), \quad u_3 = u_3(S),$$

where $u_3(S)$ is an arbitrary function; $h_2'h_3 - h_3'h_2 \neq 0$.

2. Let $F_2u_{3S} - F_3u_{2S} = 0$. In this case, a double wave exists only if Eq. (53) holds and

$$u_1 = A(S)\,h_1(p), \quad u_i = u_i(S) \quad (i = 2, 3)$$

where $u_3(S)$ and $u_2(S)$ are arbitrary functions.

Assume now that $k_3 \neq 0$. Integrating (50), we have $G = \lambda(k_2/k_3) + \Phi(p, S)$. Substituting the latter into (51) and decomposing it with respect to λ we obtain

$$k_3k_{2p} - k_2k_{3p} + (k_2F_2' + k_3F_3')(F_2u_{3S} - F_3u_{2S})(u_{1p}/\tau)_S = 0; \tag{54}$$

$$\Phi_p + (F_0' - F_2'\Phi)\left(\frac{u_{2S}}{\omega_S} - \frac{u_{3S}}{\omega_S}\frac{k_2}{k_3}\right) = 0. \tag{55}$$

Differentiating the relation $\tau + \omega u_{1p} = 0$ with respect to p and using the condition $H = 0$, we get

$$u_2F_2 + u_3F_3 = F_4 - u_1 - \tau/u_{1p}, \quad u_2F_2' + u_3F_3' = F_4' + \tau u_{1pp}/u_{1p}^2. \tag{56}$$

In this case, for $F_2F_3' - F_2'F_3 = 0$, without loss of generality, it is assumed that $F_2 = F_3 = 0$, i.e., $u_2(S)$ and $u_3(S)$ are arbitrary functions, and $u_1(p, S)$ and $\tau(p, S)$ is determined from the equations

$$\tau = (F_4 - u_1)\,u_{1p}, \quad u_{1pp}(F_4 - u_1) + F_4'u_{1p} = 0.$$

With $F_2F_3' - F_2'F_3 \neq 0$, Eqs. (56) gives

$$u_2 = [F_3'(F_4 - u_1 - \tau/u_{1p}) - F_3(F_4' + \tau u_{1pp}/u_{1p})]/(F_2F_3' - F_2'F_3),$$
$$u_3 = [-F_2'(F_4 - u_1 - \tau/u_{1p}) + F_2(F_4' + \tau u_{1pp}/u_{1p})]/(F_2F_3' - F_2'F_3). \quad (57)$$

Now, we must only satisfy Eqs. (54) and

$$\frac{\partial u_2}{\partial p} = F_2 u_{1p}, \quad \frac{\partial u_3}{\partial p} = F_3 u_{1p} \quad (58)$$

with the functions u_2 and u_3 taken from (57). According to computations,

$$F_2\left(\frac{\partial u_2}{\partial p} - F_2 u_{1p}\right) + F_3\left(\frac{\partial u_3}{\partial p} - F_3 u_{1p}\right) = 0,$$

i.e., Eqs. (58) involve only one independent equation. Thus, for the functions $\tau(p, S)$ and $u_1(p, S)$ only two equations hold.

Thus, it can be concluded that nonisentropic, nonisobaric, nonstationary, spatial double wave flows of ideal gas that are irreducible to invariant solutions with functional arbitrariness are available only in the following forms (in order of their derivation).

(1) Double waves with one arbitrary function of two arguments with the equation of state (13). In this case, $u_i = u_i(S)$ $(i = 1, 2, 3)$ are arbitrary functions of entropy. The pressure is determined from the equation $g(p) = c_1 t$ and the entropy is found from the system in involution consisting of two differential equations (14).

(2) Double waves with one arbitrary function of one argument in which $u_i = u_i(S)$ $(i = 2, 3)$ are arbitrary functions of entropy and $u_1 = u_1(p, S)$ and $\tau(p, S)$ are determined from the equations

$$\tau = u_{1p}(F_0 - u_1), \quad 2u_{1p}F_0' + u_{1pp}(F_0 - u_1) = 0 \quad (59)$$

with the arbitrary function $F_0(p)$. Excluding $u_1(p, S)$ from (59) we obtain that these double waves hold only for the equations of state satisfying $(\alpha^2 = 1)$ the relation

$$\tau\tau_{pp}[-\alpha(F_0')^3 + 3\alpha\tau_p F_0' + (\tau_p - (F_0')^2)\sqrt{(F_0')^2 - 4\tau_p}]$$
$$+ \tau\tau_p F_0''[\alpha(F_0') - 2\alpha\tau_p + F_0'\sqrt{(F_0')^2 - 4\tau_p}] + 2F_0'\tau_p^2[\alpha(F_0')^2 - 4\alpha\tau_p + F_0'\sqrt{(F_0')^2 - 4\tau_p}] = 0.$$

The functions $p = p(x_1, t)$ and $S = S(x_2, x_3, t)$ are found from the overdetermined system in involution (5), (7), and (16).

(3) Double waves with two arbitrary functions of one argument and the equation of state $\tau = g(p)A^2(S)$, in which $u_1 = h_1(p)A(S)$, and the velocity coordinates u_2 and u_3 are either of the form $u_2 = h_2(p)A(S)$, $u_3 = u_3(S)$ or $u_2 = u_2(S)$, $u_3 = u_3(S)$. In the first case the functions $h_1(p)$, $h_2(p)$, and $g(p)$ satisfy the system of two ordinary differential equations [(27), (28) or (29), (30)] $(h_1h_1'' + h_2h_2'' \neq 0)$, and $u_3(S)$ is an arbitrary function. In the second case, $u_2 = u_2(S)$, $u_3 = u_3(S)$ are arbitrary functions and $h_1(p)$ and $g(p)$ are related via the equation $g + h_1h_1' = 0$ $(h_1h_1'' \neq 0)$. In these double waves the pressure is stationary. These solutions for a polytropic gas are considered in [1–3] with the functions $u_2(S)$ and $u_3(S)$ being, however, related to $A(S)$ through the linear dependence. For our case they are arbitrary.

(4) Double waves with level straightlines with two arbitrary functions of one argument. The arbitrariness is determined by the arbitrary functions from the solution of Eq. (38). For $\tau(p, S)$ and $u_i(p, S)$ $(i = 1, 2, 3)$, there is an overedetermined system of five differential equations: (32), (39), and $H = 0$. It is rather difficult to analyze this overdetermined system for the general case of equations of state. In the particular case of the equation of state of a polytropic gas with the polytropic exponent $\gamma = 2$ and with the additional condition $\tau + u_\alpha u_{\alpha p} = 0$ (which corresponds to stationary flows) this system is compatible and has a solution with one arbitrary function of one argument.

(5) Double waves with one arbitrary function of two arguments. The arbitrariness is determined by the arbirary function from the solution of Eq. (51). The pressure in these flows is stationary. The functions

$g(p)$ and $h_i(p)$ $(i = 1, 2, 3)$ are related through Eq. (53), where $\tau = g(p) A^2(S)$ and $u_1 = h_1(p) A(S)$ and the other velocity coordinates are either $u_i = h_i(p) A(S)$ $(i = 2, 3)$ $(h_2' h_3 - h_2 h_3' \neq 0)$ or $u_2 = h_2(p) A(S)$; $u_3 = u_3(S)$ are arbitrary function or $u_2 = u_2(S)$ and $u_3 = u_3(S)$ are arbitary functions.

(6) Double waves with one arbitrary function of one argument, which is an arbitrary function in the solution of Eq. (55). The functions $\tau(p, S)$ and $u_i(p, S)$ $(i = 1, 2, 3)$ are determined as follows: either $u_2 = u_2(S)$ and $u_3 = u_3(S)$ are arbitrary functions, $u_1 = F_0 - \alpha\tau/\sqrt{-\tau_p}$, and for $\tau = \tau(p, S)$ we have

$$2\alpha F_0' \tau_p \sqrt{-\tau_p} + \tau\tau_{pp} = 0$$

where $\alpha^2 = 1$, F_0 is an arbitrary function, or u_2 and u_3 are determined by relations (57), and $\tau = \tau(p, S)$ and $u_1 = u_1(p, S)$ are found from two equations: Eq. (54) and the equation of system (58).

Thus, the validity has been established.

Theorem. *There are only six types of nonisentropic nonisobaric nonstationary spatial double wave flows of ideal gas that are irreducible to invariant solutions with functional arbitrariness.*

This work was supported by the Russian Foundation for Fundamental Research (Grant 93-013-17361).

REFERENCES

1. E. N. Zubov, "On the class of exact solutions of the equations of gas dynamics with variable entropy," in: *Numerical and Analytical Methods of Solving the Problems on the Mechanics of a Continuous Medium* [in Russian], Sverdlovsk (1981), pp. 41-67.
2. E. N. Zubov, "On double waves of the equations of gas dynamics for spatial nonstationary flows of an ideal gas with variable entropy," *Dokl. Akad. Nauk SSSR*, **263**, No. 5, 1087-1091 (1981).
3. E. N. Zubov, "On some exact solutions of the equations of gas dynamics for spatial nonstationary flows of ideal gas with variable entropy," in: *Exact and Approximate Methods to Study the Probelms of the Mechanics of a Continuous Medium* [in Russian], Sverdlovsk (1983), pp. 53-68.
4. L. V. Komarovskii, "On an exact solution of the equation of spatial unsteady gas flow of a double wave type," *Dokl. Akad. Nauk SSSR*, **135**, No. 1, 33-35 (1960).
5. L. V. Komarovsky, "On spatial gas flows with a degenerate hodograph," *Prikl. Mat. Mekh.*, **24**, No. 3, 491-495 (1960).
6. L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Nauka, Moscow (1978).